MATERI PERSIAPAN KUIS 1 MATEMATIKA INFORMATIKA

RUANG VEKTOR - VEKTOR (M1-M6)

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- **What is Vector?**

  Vector is an object that has both a magnitude and a direction. So Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction.

- **Vector Space**

  A vector space is a set of vectors that can be added together and multiplied by scalars (real numbers) while satisfying certain axioms (e.g., associativity, distributivity, zero vector existence). A set V with addition and scalar multiplication operations satisfying 10 axioms

  Key Materials :

  - Contains zero vector 0
  - Close under addition: If $u,v \in V$, then $u + v \in V$
  - Closed under scalar multiplication: If $v \in V$, $k \in \mathbb{R}$, then $kv \in V$

  Example :

  - $\mathbb{R}^n$ = all n-dimensional real vectors
  - M_{nn} = all n×n matrices
  - P_n = all polynomials of degree $\leq n$

- **Linear Independence**

  Vectors $\{v_1,...,v_n\}$ are linearly independent if the only solution to:

  $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$

  is $c_1 = c_2 = ... = c_n = 0$.

  Key Materials :

  - Linear dependence implies at least one vector can be written as a linear combination of others.
  - Linear independence means no vector can be formed from the others.

  How to check :

  - Create a matrix with the vectors as columns
  - Perform row reduction (Gaussian elimination)
  - If only trivial solution → independent, If non-trivial solutions exist → dependent

- **Linear Combination**

  A vector w is a linear combination of vectors $v_1, v_2, ..., v_n$ if:

  $w = c_1v_1 + c_2v_2 + ... + c_nv_n$ for some scalars $c_1,...,c_n \in \mathbb{R}$

  Example:

  Is [2 5 8] a linear combination of [1 0 1], [1 1 1], [0 2 3]?

  → Solve the system: $c_1[1\ 0\ 1] + c_2[1\ 1\ 1] + c_3[0\ 2\ 3] = [2\ 5\ 8]$

- **Proving a Linear Combination**

  To prove one vector is a linear combination of others, solve:

  $w = c_1v_1 + c_2v_2 + ... + c_nv_n$

  Example:

  Prove [5 2 4] is a combination of [1 0 1] and [2 1 1]

  → $c_1[1\ 0\ 1] + c_2[2\ 1\ 1] = [5\ 2\ 4]$ → Solve the system

- **Exercises and Answers**

  Q1. Determine if vectors [1 2] and [3 6] are linearly independent

  → They are dependent (second is a scalar multiple of the first)

  Q2. Is [4 5] a linear combination of [1 2], [1 1]?

  Solve:

  $c_1[1\ 2] + c_2[1\ 1] = [4\ 5]$

  → $c_1 + c_2 = 4$, $2c_1 + c_2 = 5$ → $c_1 = 1$, $c_2 = 3$ → Yes

Vector Space – Exercises

Question

Q1. Is the set of all 2x2 matrices a vector space?

Q2. Is the set of all vectors in $\mathbb{R}^3$ with zero third component a vector space?

Q3. Does the set of all polynomials of degree exactly 2 form a vector space?

Q4. Is the set of all continuous functions a vector space?

Q5. Is $\mathbb{R}^2$ under standard addition and scalar multiplication a vector space?

Q6. Show that the zero vector is unique in any vector space.

Q7. Prove that every vector has a unique additive inverse in a vector space.

Q8. Does closure under addition imply associativity? Explain.

Q9. Is the union of two subspaces always a subspace? Justify.

Q10. Is the set of all 2D vectors with integer entries a vector space over $\mathbb{R}$?

Vector Space – Answers and Explanations

Answers

Q1. Yes, 2x2 matrices form a vector space under standard addition and scalar multiplication.
Explanation: They satisfy all 10 axioms of a vector space, including closure, associativity, identity, and inverses.
Q2. Yes, because they are closed under vector addition and scalar multiplication.
Explanation: The set is a subspace of $\mathbb{R}^3$.
Q3. No, because it's not closed under addition (sum of two degree 2 polynomials might be degree ≤ 2).
Explanation: Closure under addition fails since degree may drop.
Q4. Yes, continuous functions form an infinite-dimensional vector space.
Explanation: Operations like addition and scalar multiplication preserve continuity.
Q5. Yes, $\mathbb{R}^2$ is a standard example of a vector space.
Explanation: All vector space axioms are satisfied.
Q6. Yes, and it's proved by assuming two zero vectors and showing they are equal.
Explanation: Let 0 and 0' be zero vectors, then 0 + 0' = 0' implies 0 = 0'.
Q7. Yes, every vector has a unique additive inverse in a vector space.
Explanation: Let v + w = 0 and v + w' = 0, then w = w'.
Q8. No, closure does not imply associativity; associativity is a separate axiom.
Explanation: Each axiom must be independently verified.
Q9. No, the union of two subspaces is not necessarily a subspace.
Explanation: Counterexample: x-axis and y-axis in $\mathbb{R}^2$.
Q10. No, because integers are not closed under scalar multiplication by real numbers.
Explanation: For example, 0.5 * [1 0] = [0.5 0], which is not in $\mathbb{Z}^2$.

Linear Independence – Exercises

Question

Q1. Are [1 0], [0 1] linearly independent?
Q2. Are [2 4], [1 2] linearly independent?
Q3. Determine if [1 2 3], [4 5 6], [7 8 9] are linearly independent.
Q4. Can three vectors in $\mathbb{R}^2$ be linearly independent?
Q5. Can the zero vector be part of a linearly independent set?
Q6. If $\{v_1, v_2\}$ is linearly dependent, what does it mean?
Q7. Are the columns of the identity matrix linearly independent?
Q8. Determine if vectors [1 -1 0], [2 1 1], [3 0 1] are dependent.

Q9. Show that any set with more vectors than the dimension is dependent.
Q10. Are [1 1 0], [2 2 0], [0 0 1] independent?

Linear Independence - Answers and Explanations

Answer
Q1. Yes, they are independent.
Explanation: Each cannot be written as a scalar multiple of the other.
Q2. No, because [2 4] = 2 * [1 2].
Explanation: They are linearly dependent.
Q3. No, determinant is 0 → dependent.
Explanation: The rows are linearly dependent.
Q4. No, maximum number of independent vectors in $\mathbb{R}^2$ is 2.
Explanation: More than 2 vectors in $\mathbb{R}^2$ are always dependent.
Q5. No, a set containing the zero vector is always dependent.
Explanation: c=1 for zero vector gives non-trivial solution.
Q6. It means at least one vector is a scalar multiple of the other.
Explanation: That's the definition of dependence for two vectors.
Q7. Yes, because each standard basis vector is independent.
Explanation: Each has a unique 1 in one coordinate.
Q8. Yes, they are linearly dependent.
Explanation: Row reduction yields a free variable.
Q9. Yes, it's a fundamental theorem in linear algebra.
Explanation: In $\mathbb{R}^n$, any set of >n vectors is dependent.
Q10. No, because [2 2 0] = 2 * [1 1 0]
Explanation: The first two vectors are dependent, so the whole set is.

Linear Combination - Additional Exercises
Q1. Is [3 4] a linear combination of [1 0], [0 1]?
Q2. Can [2 2] be written as a combination of [1 1] and [1 -1]?
Q3. Write [2 3 4] as a combination of [1 0 0], [0 1 0], [0 0 1].
Q4. Express [5 2 4] using [1 0 1] and [2 1 1].
Q5. Is [4 4] in the span of {[1 2], [2 1]}?
Q6. Determine c_1, c_2 such that c_1[1 1] + c_2[2 1] = [5 3]
Q7. Can [6 8] be written as a combination of [2 3] and [1 2]?
Q8. Find a combination of [2 1 0] and [1 2 0] that gives [5 4 0].
Q9. Are [3 2], [1 0] sufficient to express [4 2]?
Q10. Use linear combination to determine if [5 7] is in span{[1 1], [2 3]}.

Linear Combination - Answers and Explanations

Answer

Q1. Yes, because [3 4] = 3*[1 0] + 4*[0 1]

Explanation: It is a combination of standard basis vectors.

Q2. Yes, [2 2] = 1*[1 1] + 0*[1 -1]

Explanation: Solve system of equations.

Q3. Yes, just use coefficients as [2 3 4] = 2*[1 0 0] + 3*[0 1 0] + 4*[0 0 1]

Explanation: Standard basis expansion.

Q4. Yes, solution: [5 2 4] = 1*[1 0 1] + 2*[2 1 1]

Explanation: Solving the linear system gives coefficients.

Q5. Yes, solve: a*[1 2] + b*[2 1] = [4 4]

Explanation: It has a solution (a=0.8, b=1.6).

Q6. c_1 = 1, c_2 = 2

Explanation: 1*[1 1] + 2*[2 1] = [5 3]

Q7. Yes, use system of equations to verify.

Explanation: Solution exists: a=2, b=0.

Q8. a = 1, b = 1

Explanation: 1*[2 1 0] + 1*[1 2 0] = [3 3 0]

Q9. Yes, a = 1, b = 1

Explanation: [3 2] = [1 0] + [2 2]

Q10. Yes, [5 7] = 1*[1 1] + 2*[2 3]

Explanation: Check by substitution.

Proving a Linear Combination - Additional Exercises

Question

Q1. Prove [3 5] is a combination of [1 2] and [2 1].

Q2. Show [7 8 9] can be formed from [1 0 1], [1 1 1], [0 2 3].

Q3. Is [5 2 4] a combination of [1 0 1] and [2 1 1]?

Q4. Demonstrate that [4 4] is a combination of [1 2] and [2 1].

Q5. Prove [0 0 1] is not in the span of {[1 0 0], [0 1 0]}

Q6. Show that [6 7] is a combination of [2 3] and [1 2]

Q7. Is [3 6] in the span of {[1 2], [1 1]}?

Q8. Show how [4 6] can be expressed from [2 3] and [1 1]

Q9. Find scalars a, b so that a[1 1] + b[2 1] = [5 3]

Q10. Can [10 10] be written as a combination of [2 3] and [3 2]?

Proving a Linear Combination - Answers and Explanations

Answer

Q1. Yes, [3 5] = 1*[1 2] + 1*[2 1]
Explanation: 1*[1 2] + 1*[2 1] = [3 3]
Q2. Yes, [7 8 9] = 2*[1 0 1] + 3*[1 1 1] + 1*[0 2 3]
Explanation: Solve the system.
Q3. Yes, [5 2 4] = 1*[1 0 1] + 2*[2 1 1]
Explanation: Same as previous example.
Q4. Yes, a = 0.8, b = 1.6
Explanation: Satisfies linear combination equation.
Q5. No, [0 0 1] cannot be formed by [1 0 0] and [0 1 0]
Explanation: Missing third dimension.
Q6. Yes, a=2, b=1
Explanation: [6 7] = 2*[2 3] + 1*[1 2]
Q7. Yes, a=3, b=0
Explanation: [3 6] = 3*[1 2] + 0*[1 1]
Q8. Yes, a=1, b=2
Explanation: [4 6] = 1*[2 3] + 2*[1 1]
Q9. a=1, b=2
Explanation: Check: 1*[1 1] + 2*[2 1] = [5 3]
Q10. Yes, a=2, b=2
Explanation: 2*[2 3] + 2*[3 2] = [10 10]